Math 4200-001
Homework 11
4.1-4.2
Due Wednesday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)

Math 4200
Monday November 9

4.2 The Residue Theorems!

Announcements: I'll move HW due date to usual Friday time.
But §4.1-4.2 material is on Friday midterm.
Some time on Wed for HW questions
(Residue table on exam.)
Review notes in today's notes → we'll talk about on Wed.

will be given on the exam.

the residue at z_0, $\mathrm{Res}(g/h; z_0)$ *is given by*

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times$$
$$\begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z)=0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=0,$ $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0)=0=h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)}-\frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0)=0, g'(z_0)\neq 0,$ $h(z_0)=0=h'(z_0)$ $=h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)}-\frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z)=(z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z)=(z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0)=$ $\ldots=h^{k-1}(z_0)$ $=0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

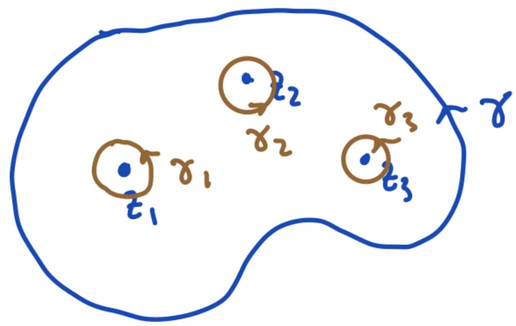
Review.

<u>Residue Theorem 1</u> (Replacement Theorem version, from Friday): Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$ Let γ be a simple closed contour in A which contains none of the singularities, and which bounds a a subregion B containing some of the singularities, in the counterclockwise direction. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_j \in B} Res(f, z_j).$$

Short proof

Replacement Thm



$$\int_\gamma f(z)dz = \sum_{z_j \in B} \oint_{\underbrace{|z-z_j| = r_j}} f(z)dz$$

near z_j $\quad f = S_{j1} + S_{j2}$

$$\sum_{n=0}^{\infty} a_{jn}(z-z_j)^n + \underbrace{\sum_{m=1}^{\infty} b_{jm}(z-z_j)^{-m}}$$

$R>0$ $\quad$ converges in $\mathbb{C} \setminus \{z_j\}$.

$$\boxed{\int (z-z_0)^k\,dz = \frac{(z-z_0)^{k+1}}{k+1} + C \quad \forall k \in \mathbb{Z} \text{ except } k=-1}$$

unif conv

$\Rightarrow$ swap $\int, \Sigma$

FTC $\Rightarrow$ only one non-zero term

$b_{j,-1} \cdot 2\pi i = 2\pi i\, Res(f; z_j)$

Residue Theorem 2 (Deformation Theorem version.) Let Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \dots z_k\} \subseteq A$. Let γ be a closed curve which is homotopic to a point in A. Then

$$\int_\gamma f(z)\, dz = 2\,\pi\, i \sum_{j=1}^{k} Res(f, z_j)\; I(\gamma; z_j)$$

in thm 1, $I = 1$ for each z_j inside γ

For each isolated singularity z_j we have the Laurent series

- $$S_j(z) = S_{j1}(z) + S_{j2}(z) = \sum_{n=0}^{\infty} a_{jn}(z - z_j)^n + \sum_{m=1}^{\infty} \frac{b_{jm}}{(z - z_j)^m}.$$

Because the z_j are point singularities, the singular part of the series, $S_{j2}(z)$ converges in $\mathbb{C} \setminus \{z_j\}$, and the non-singular part converges for $0 \le |z - z_j| < R_j$ for some positive radius of convergence R_j.

Now consider

- $$g(z) := f(z) - \sum_{j=1}^{k} S_{j2}(z).$$ converges in $A \setminus \{z_1, z_2, \dots z_N\}$.

1

Explain why $g(z)$ has removable singularities at each z_j, so can be treated as being analytic in A.

$f(z) - S_{j2}(z)$ has remov. sing @ z_j because it equals $S_{j1}(z)$ there.

So $g(z) = f(z) - S_{j2}(z) - \sum_{\ell \neq j} S_{\ell 2}(z)$

So $g(z)$ extends to be analytic @ z_j.

Deformation Thm

$$\Longrightarrow \int_\gamma g(z)\,dz = 0$$

$$\Longrightarrow \int_\gamma f(z)\,dz = \int \sum_{j=1}^{k} S_{j2}(z)\,dz$$

$$= \sum_{j=1}^{k} \sum_{m=1}^{\infty} \int_\gamma b_{jm}(z - z_j)^{-m}\,dz$$

$$= \sum_{j=1}^{k} Res(f; z_j) \underbrace{\int_\gamma \frac{1}{z - z_j}\,dz}_{2\pi i\, I(\gamma; z_j)}$$

Thus we may consider g to be analytic in A, so since γ is homotopic as closed curves to a point in A,

$$\int_{\gamma} g(z)\ dz = 0.$$

Expand this to get the result!

Residue Theorem for exterior domains (This is our 3rd residue theorem). Let γ be a simple closed contour enclosing a region A, oriented counterclockwise as usual. Let K be a compact subset in A (possibly empty), and let $\{z_1, z_2, \ldots z_n\}$ be points exterior to γ. Let

$$f: \mathbb{C} \setminus \{K \cup \{z_1, z_2, \ldots z_n\}\} \to \mathbb{C}$$

be analytic. Then

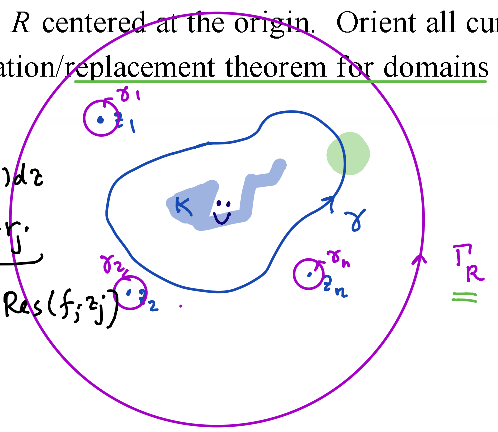
$$\int_\gamma f(z)dz = -2\pi i \left(Res(f; \infty) + \sum_{k=1}^{n} Res(f; z_k) \right)$$

where

$$Res(f; \infty) := Res\left(-\frac{1}{z^2} f\left(\frac{1}{z}\right); 0 \right).$$

proof: Enclose γ and all of the singularities $\{z_1, z_2, \ldots z_n\}$ in a large disk of radius R centered at the origin as pictured, and let γ_j be a circle concentric with the singularity z_j and of sufficiently small radius r_j so that the closed disks they enclose are disjoint, and don't intersect γ or Γ_R, the circle of radius R centered at the origin. Orient all curves counterclockwise. Then apply the deformation/replacement theorem for domains with holes.

$$\oint_{|z|=R} f(z)dz = \oint_\gamma f(z)dz + \underbrace{\sum_{j=1}^{n} \oint_{|z-z_j|=r_j} f(z)dz}_{2\pi i \sum_j Res(f; z_j)}$$



You will arrive at an equation which is equivalent to

$$\int_\gamma f(z)dz = -2\pi i \sum_{k=1}^{n} Res(f; z_k) + \int_{\Gamma_R} f(z)\, dz.$$

To evaluate the contour integral over Γ_R do an analytic change of variables,

$\zeta = \frac{1}{z}$, $z = \frac{1}{\zeta}$ which will give you a contour integral over a circle of radius $\frac{1}{R}$, traversed clockwise. Evaluate this integral with the earlier versions of the Residue

Theorems and the result will follow. Analytic change of variables for contour integrals is justified on the next page.

$$\oint_{|z|=R} f(z)\,dz = \int_0^{2\pi} f(Re^{i\theta})\,Rie^{i\theta}\,d\theta.$$

$z = Re^{i\theta}$

??

$z = \frac{1}{\zeta}$ $\quad$ $|z| = R,\ |\zeta| = \frac{1}{R}$

$dz = -\frac{1}{\zeta^2}\,d\zeta$

$z = Re^{i\theta}$

$\zeta = \frac{1}{z} = \frac{1}{R}e^{-i\theta}$

$$\oint_{|\zeta|=\frac{1}{R}} f\left(\tfrac{1}{\zeta}\right)\left(-\tfrac{1}{\zeta^2}\right)d\zeta$$

$f\left(\frac{1}{\zeta}\right)$ is analytic inside $|\zeta| = \frac{1}{R}$ except maybe at $\zeta = 0$

$$= -2\pi i \underbrace{\mathrm{Res}\left(f\left(\tfrac{1}{\zeta}\right)\left(-\tfrac{1}{\zeta^2}\right); 0\right)}_{:= \mathrm{Res}(f; \infty)}$$

Index is -1

$\blacksquare$

<u>Theorem</u> Analytic change of variables in contour integrals: Consider the contour integral

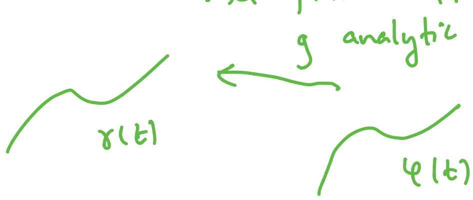
$$\int_\gamma f(z)\,dz.$$

Suppose there is an invertible analytic function g with range that includes γ, $z = g(\zeta)$, $\zeta = g^{-1}(z)$. Then the formal substitution $z = g(\zeta)$, $dz = g'(\zeta)\,d\zeta$ yields an equality of integrals

$$\int_\gamma f(z)\,dz = \int_{g^{-1}(\gamma)} f(g(\zeta))\, g'(\zeta)\, d\zeta.$$

proof: Let $\gamma : [a, b] \to \mathbb{C}$ be a parameterization of the contour on the left. Write $\varphi(t) = g^{-1}(\gamma(t))$ to parameterize the contour on the right. (Assume γ is C^1 rather than piecewise C^1 for simplicity). Compute both contour integrals and use the chain rule for curves to verify that the integrals agree.

$\gamma = g \circ \varphi$ $\qquad$ I'll fill in after class.



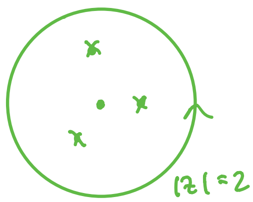
Example of Residue Theorem for exterior domains: Compute

$$-2\pi i \, Res(f;\infty) = \int_\gamma \frac{3z^2+7}{z^3+2z-3}\,dz$$

$z=1$ is root of denom.

where γ is the circle $|z| = 2$, oriented counter-clockwise as usual. First verify that all of roots of the cubic denominator lie inside the circle, so we'll only need the residue at ∞,

- $Res(f;\ \infty) := Res\left(-\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right).$



$$|z^3+2z-3| \geq \big|\, |z^3| - |2z-3| \,\big|$$
$$\geq \big|\, |z|^3 - |2z| - 3 \,\big|$$

if $|z| \geq 2$ consider

$p(t) = t^3-2t-3$

$p(2) = 8-4-3 = 1$

$p'(t) = 3t^2-2 \geq 10$ for $t \geq 2$.

so by reverse Δ ineq., $|z^3+2z-3| \geq 1$ $|z| \geq 2$.

$Res(f;\infty)$

$$= Res\left(-\frac{1}{z^2}\ \frac{3\frac{1}{z^2}+7}{\frac{1}{z^3}+\frac{2}{z}-3};0\right)$$

$$= Res\left(-\frac{\frac{3}{z^2}+7}{\frac{1}{z}+2z-3z^2};0\right)$$

$$= Res\left(-\frac{\frac{3}{z}+7}{1+(\underbrace{2z^2-3z^3}_{=-w})};0\right)$$

for this Residue just need Laurent series near isolated sing $z=0$

$$\left(-\frac{3}{z}-7\right)\left(1+\underset{\uparrow\ -2z^2+3z^3}{w}+w^2+\dots\right)$$

$$= -\frac{3}{z} + \text{analytic}$$

$Res = -3.$

final ans is $(-2\pi i)(-3) = 6\pi i$

$$-\frac{3}{z}\left[\frac{1+\frac{7z}{3}}{1+\text{stuff}}\right]$$

$\uparrow$ analytic & $=1$ @ $z=0$

Wed: HW ?'s

Review notes

Review notes for exam this Friday November 13.

Exam will cover 2.4-2.5, 3.1-3.3, 4.1-4.2, and implicitly use the earlier course material.

email me (korevaar "at" math.utah.edu) your preferred two hour time slot on Friday, starting on the hour between 10:00 a.m. and 4:00 p.m.

I will email you a .pdf of the exam at the start time or a few minutes early. Unless you tell me otherwise I'll email it to the return address of the email you send me.

Complete the exam and upload a .pdf of your solutions to Gradescope by two hours after your start time. For insurance or if you have trouble uploading, email me a .pdf within your time limit as well.

The exam is closed book, closed notes, closed internet etc. Your only resource is yourself. I'll ask you to sign an honor-code like statement on the front of your exam which will be part of what you upload to Gradescope.

As with the first exam there will be some required problems at the beginning of the exam, and then you'll be asked to complete several substantial problems where you have some choice about which problems to tackle. There will be a mixture of theorem proofs/explanations, along with computations.

There is a practice exam posted in our CANVAS notes, and I'll go over it on Thursday during office hours, starting at 2:00.

Topics:

2.4 Cauchy integral formula

Index $I(\gamma; z_0)$

C.I.F. for closed contour γ contractible in a domain on which $f(z)$ is analytic.

formulas and estimates for derivatives

Liouville's Theorem

Fundamental Theorem of Algebra

Morera's Theorem.

2.5 Maximum modulus principle and harmonic functions

Mean value property for $f(z)$ analytic

Clever proof for harmonic conjugates in simply connected domains

Mean value property for harmonic functions

Maximum modulus principle for $f(z)$ analytic

Maximum and minimum principles for harmonic functions

3.1 Convergent sequences and series of analytic functions

why uniform limits of analytic functions are analytic, and why the derivative of the limit of analytic functions is the limit of the derivatives

Weierstrass M test

3.2 Power series and Taylor's Theorem

radius of convergence

term by term differentiation

uniqueness

analytic if and only if power series

isolated zeroes theorem

multiplication of power series

key examples

3.3 Laurent series

analytic in an annulus (including punctered disk case) if and only if Laurent series.... .where does each piece converge?

uniqueness

isolated zeroes classification

residue

multiplication of Laurent series.

geometric series wizardry.

4.1 Calculating residues at isolated singularities

$$f(z) = \frac{f_1(z)}{(z - z_0)^k} + f_2(z)$$

$$f(z) = \frac{g(z)}{h(z)} = \frac{\sum_{n=M}^{\infty} a_n (z - z_0)^n}{\sum_{n=N}^{\infty} \tilde{a}_n (z - z_0)^n}$$

simple poles

table will be provided

4.2 Residue theorem

statement and proof for γ contractible in A via the deformation theorem

statement and proof if γ is a simple closed curve bounding a domain via Replacement Theorem for domains with holes.

contour integral computations via the residue theorems

residues at ∞ (I'll remind you of the formula if you have to use it.)